ON THE DYNAMICS OF A PARTICLE INSIDE A U – SHAPED ROTATING THIN TUBE

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ABSTRACT

In this paper the problem of the strongly nonlinear motion of a particle on a rotating parabola is generalized for an arbitrary U-shaped curve. The governing equation of motion is deducted and then particularized on three cases, namely quadratic parabola, quartic parabola and cosine curve. Each case is numerically investigated for various small and large parameters and the results are contrasted with those provided by a relatively new analytical technique called energy balance method. The importance of the relative equilibrium points on the particle's dynamics is highlighted.

Keywords: *particle on a rotating U-shaped curve, energy balance method, numerical analysis.*

1. INTRODUCTION

Highly nonlinear oscillators are extremely present in the engineering applications. Unfortunately, such problems do not allow for an exact solution so the researcher interested in solving them has to resort to numerical or analytical approximate solutions.

In the recent years, many new methods have appeared in the literature to estimate the behaviour of such oscillators. One of these approaches is the energy balance method, proposed by Chinese mathematician He [1 – 3]. It gives the frequency-amplitude relationship for a single-degree-of-freedom conservative nonlinear oscillator. The procedure is quite straightforward: first a variational principle is established, then the Hamiltonian is constructed and, finally, the above-mentioned relationship is obtained by collocation. The reported results for different oscillators show that the method provides excellent or, at least, reasonably accuracy [4 – 6].

A complicated but interesting highly nonlinear problem is represented by the motion of a particle on a rotating parabola. First discussed by Nayfeh [7], it was approximately solved by various researchers including Ganji et al [4] with energy balance method, Hatami et al [8] with multi-step differential transform method, Baleanu et al [9] with fractional Lagrangian and Mirzabeigy et al [10] with other analytical and semi-analytical techniques.

In this paper, we extend the topic by considering instead the quadratic parabola an arbitrary U-shaped rotating curve. The problem is formulated in Section 2 while in the Appendix the equation of motion is exhaustively deducted. The general equation is customized in Section 3 and the particular cases are analysed numerically and analytically in Section 4. Finally, paper concludes in Section 5.

2. PROBLEM FORMULATION

A rigid frictionless thin tube having the shape of a U-shaped curve $y = f(x)$ is forced to rotate with constant angular frequency Ω around the *Oy* axis of a Cartesian frame *Oxy*. As the tube rotates, a small ball of mass *m* moves inside it (see Figure 1). The function *f* is assumed to be continuous and to verify the conditions $f(0) = f'(0) = 0$ and $f''(0) > 0$.

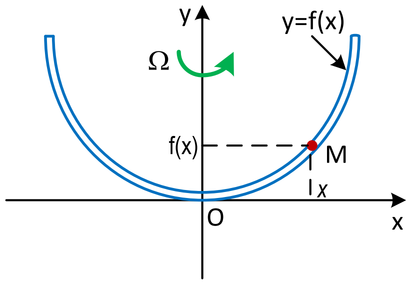

Figure 1. A small ball moving inside a rotating rigid thin tube

The particle's equation of relative motion is

$$\left[1+\left(f'(x)\right)^2\right]\ddot{x}+f'(x)\,f''(x)\,\dot{x}^2-\Omega^2 x+g\,f'(x)=0 \quad (1)$$

where the dots denote the differentiation with respect to time *t* (see Appendix for details). The initial conditions were chosen as

$$x(0)=A,\ \dot{x}(0)=0 \quad (2)$$

The Hamiltonian associated to the differential equation (1) is

$$H=\frac{1}{2}\left[1+\left(f'(x)\right)^2\right]\dot{x}^2-\frac{1}{2}\Omega^2x^2+g\,f(x)= \\ =g\,f(A)-\frac{1}{2}\Omega^2A^2=\text{constant} \quad (3)$$

Generally, the equation (1) describes an oscillatory motion around a relative equilibrium position x_{REP}, given by

$$g\,f'(x_{REP})=\Omega^2 x_{REP} \quad (4)$$

3. PARTICULAR CASES

In what follows we consider two particular shapes for curve $y = f(x)$:

a) *Generalized quadratic parabola* $f(x) = p\,x^{2n}$, with $p > 0$ a constant and n a positive integer.

Equations (1) and (3) are rewritten as

$$\left(1+4n^2p^2x^{4n-2}\right)\ddot{x}+4n^2(2n-1)p^2x^{4n-3}\dot{x}^2-\\-\Omega^2x+2npgx^{2n-1}=0 \tag{5}$$

$$H=\frac{1}{2}\left(1+4n^2p^2x^{4n-2}\right)\dot{x}^2-\frac{1}{2}\Omega^2x^2+pgx^{2n}=\\=pgA^{2n}-\frac{1}{2}\Omega^2A^2=\text{constant} \tag{6}$$

The relative equilibrium positions are provided by

$$x_{REP}=0 \quad \text{and} \quad x_{REP}=\pm\left(\frac{\Omega^2}{2pg}\right)^{\frac{1}{2n-2}} \tag{7}$$

In Section 4 we solve numerically the equation (5) for n = 1 and n = 2. It becomes

$$n=1: \left(1+4p^2x^2\right)\ddot{x}+4p^2x\dot{x}^2+\left(2pg-\Omega^2\right)x=0$$
$$H=\frac{1}{2}\left(1+4p^2x^2\right)\dot{x}^2+\left(pg-\frac{\Omega^2}{2}\right)x^2 \tag{8}$$
$$x_{REP}=0$$

$$n=2: \left(1+16p^2x^6\right)\ddot{x}+48p^2x^5\dot{x}^2+\left(4pgx^2-\Omega^2\right)x=0$$
$$H=\frac{1}{2}\left(1+16p^2x^6\right)\dot{x}^2+\left(pgx^2-\frac{\Omega^2}{2}\right)x^2 \tag{9}$$
$$x_{REP}=0 \quad \text{and} \quad x_{REP}=\pm\frac{\Omega}{2\sqrt{pg}}$$

b) $f(x)=p(1-\cos x),\ x\in(-\pi,\pi),\ p>0$

From (1), (3) and (4) one gets

$$\left(1+p^2\sin^2 x\right)\ddot{x}+p^2\sin x\cos x\cdot\dot{x}^2-\Omega^2x+pg\sin x=0$$
$$H=\frac{1}{2}\left(1+p^2\sin^2 x\right)\dot{x}^2-\frac{1}{2}\Omega^2x^2+pg(1-\cos x)=\\=pg(1-\cos A)-\frac{1}{2}\Omega^2A^2=\text{constant} \tag{10}$$
$$x_{REP}=\begin{cases}0, \text{if } \Omega^2\geq pg\\ 0 \text{ and } x^*, \text{if } \Omega^2<pg\end{cases}, \text{ with } \frac{\Omega^2}{pg}x^*=\sin x^*.$$

4. NUMERICAL SIMULATIONS

In this section we report a number of numerical results concerning the mechanical motion analysed before.

4.1. Quadratic parabola $f(x)=px^2$

As observed in the Introduction, this case was intensively studied by means of different approaches. Here, we just mention the *energy balance method* proposed by He [2]. In accordance with this technique, the solution of (8) is written as

$$x(t)=A\cos\omega t \tag{11}$$

where the frequency ω is approximated by

$$\omega_{EBM}=\sqrt{\frac{2pg-\Omega^2}{1+2p^2A^2}} \tag{12}$$

In fact, by applying the same method to the general case (5) one gets

$$\omega_{EBM}=\sqrt{\frac{\left(2^n-1\right)pgA^{2n-2}-2^{n-2}\Omega^2}{2^{n-2}+n^2p^2A^{4n-2}}} \tag{13}$$

To check the closeness between the numerical results, obtained with the MatLab package, and the energy balance method's results, we have carried out a number of simulations. Some of the results are displayed in Figures 2 to 3. As a general trend, for small amplitudes A and arbitrary (p,Ω) the two solutions are almost identical, the relative errors between their periods being inferior to 1% (see Figure 2). By increasing A the two solutions start to differ both in terms of periods and shapes. Relative errors of 1 – 10% between the oscillation periods are frequently observed and the form of cosine wave is deformed towards a square wave (see Figure 3). The behaviour described above remains the same for many other combinations (p,Ω), as shown in Table 1.

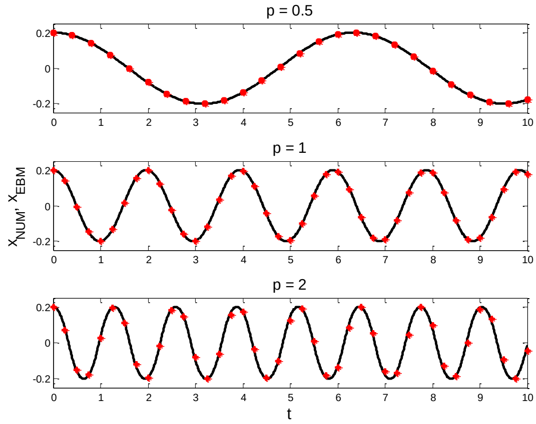

Figure 2. Comparison of analytical (red asterisks) and numerical solutions (continuous black curve) for $A=0.2$, $\Omega=3$ and different p.

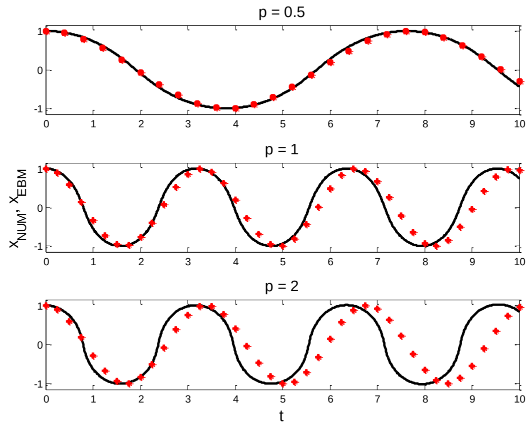

Figure 3. Comparison of analytical (red asterisks) and numerical solutions (continuous black curve) for $A=1$, $\Omega=3$ and different p.

Table 1. The relative errors (%) between the oscillation periods T_{NUM} and T_{EBM} for different *p*, *A* and Ω

p = 0.5				
Ω	*A* = 0.2	*A* = 0.5	*A* = 1	*A* = 2
1.0	0.028	0.099	0.821	3.277
1.5	0.017	0.151	0.802	3.167
2.0	0.004	0.199	0.773	3.294
2.5	0.006	0.148	0.743	3.182
3.0	0.030	0.117	0.678	3.098
p = 1.0				
Ω	*A* = 0.2	*A* = 0.5	*A* = 1	*A* = 2
1.0	0.053	0.811	3.071	6.198
2.0	0.074	0.765	3.053	6.317
3.0	0.081	0.743	3.082	6.401
4.0	0.074	0.796	4.280	6.634
4.4	0.053	0.774	3.151	6.532
p = 2.0				
Ω	*A* = 0.2	*A* = 0.5	*A* = 1	*A* = 2
1.0	0.504	3.106	6.748	8.895
3.0	0.418	2.713	6.757	9.385
5.0	0.415	3.298	6.625	8.606
6.0	0.392	3.284	6.507	9.247
6.3	0.339	3.334	6.499	9.217

Three parameters, *p*, *A* and Ω, are involved in the expression (12). If *p* and *A* are maintained fixed, then an increase of angular frequency Ω leads to a growth of period $T_{EBM} = 2\pi/\omega_{EBM}$ (as well as of T_{NUM}). The relative error $\varepsilon = \left|\frac{T_{NUM} - T_{EBM}}{T_{NUM}}\right| \cdot 100\%$ is only to a small extend influenced by changing the parameter Ω. For fixed A and Ω, the oscillation period T_{EBM} decreases with p for $p < p^* = \frac{\Omega^2 A + \sqrt{\Omega^4 A^2 + 2g^2}}{2gA}$ and increases otherwise. The relative error is proportional to *p*. Finally, if only *A* is varied, then T_{EBM}, T_{NUM} and ε have the same trend with the amplitude *A*.

The energy balance method provides the expression (12) for oscillation frequency ω_{EBM} only if $\Omega < \Omega^* = \sqrt{2pg}$. If Ω is just a little less than Ω^*, the period T_{EBM} reaches high values but remains extremely close by T_{NUM} for small *A*. As an example, if *p* = 1 and $A = 0.2$, then for $\Omega = 4.472 < \Omega^* = 4.472136$, one gets $T_{EBM} = 187.251$ *s* and T_{NUM} = 187.145 *s*. The two values differ by approximately 10% for *p* = *A* = 2, $\Omega = 6.324 < \Omega^* = 6.324390$, when $T_{EBM} = 430.669$ *s* and T_{NUM} = 390.510 *s* (see Figure 4). If Ω exceeds Ω^* then both solutions diverge to infinity, meaning that the ball ceases to oscillate around $x_{REP} = 0$ and tends to leave the tube by one of its ends (depending on its initial position).

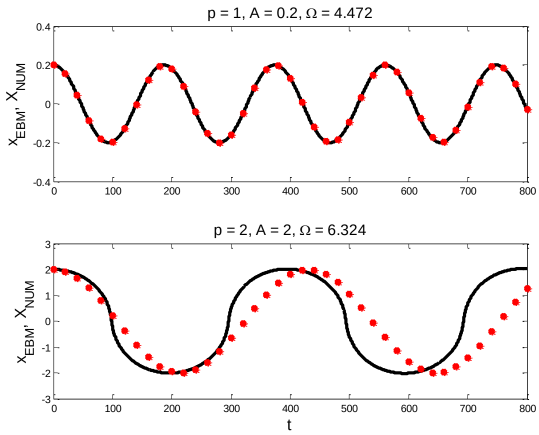

Figure 4. Comparison of analytical (red asterisks) and numerical solutions (continuous black curve) close to the upper limit of Ω.

4.2. Quartic parabola $f(x) = p\,x^4$

If $n \geq 2$, the expression (13) represents a reasonable approximation for the oscillation's frequency only for small Ω, no matter *p* and *A* are. Moreover, starting with some Ω the particle will move the oscillation centre from $x_{REP} = 0$ to one of the other positions for x_{REP} (depending on the sign of *A*). Let us select *n* = 2 (quartic parabola), *p* = 0.5, *A* = 0.2 and analyse what happens if Ω is gradually increased. From (13), one gets

$$\omega_{EBM} = \sqrt{\frac{3pgA^2 - \Omega^2}{1 + 4p^2A^6}} = \sqrt{\frac{0.6 - \Omega^2}{1.000064}} \qquad (13)$$

so the energy balance method provides an approximate solution for (9) only if $\Omega \leq \sqrt{0.6} = 0.7746$. For $\Omega \leq 0.5$ the error ε is less than 6 % (see the upper panel in Figure 5).

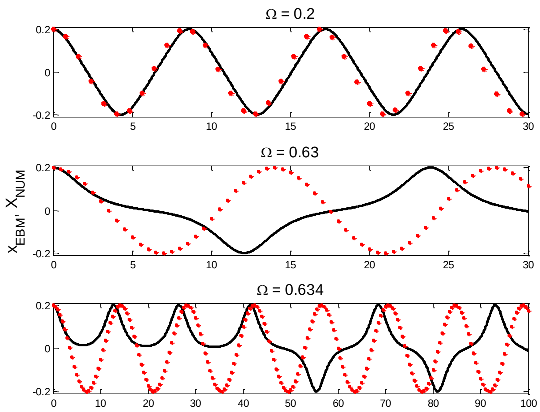

Figure 5. Comparison of analytical (red asterisks) and numerical solutions (continuous black curve) for quartic parabola $f(x) = p\,x^4$ with *p* = 0.5, *A* = 0.2 and different rotation frequencies Ω.

As from $\Omega = 0.5$ the aspect of the numerical solution is noticeably removed from the sinusoidal shape and the error ε gets unacceptable values (40 % for $\Omega = 0.63$), as shown in the middle panel of Figure 5.

Around $\Omega = 0.6325$ the numerical solution shows the first signs that the centre of oscillation will change (the lower panel in Figure 5). This will happen quite quickly (around $\Omega = 0.636$), and increased values for Ω will lead to completely different numerical and analytical solutions (e.g. if $\Omega = 0.77, T_{EBM} = 74.57$ while $T_{NUM} = 5.91$).

Figure 6 exhibits the three situations in which the numerical solution can be found for even higher values of Ω. Thus, if $\Omega < \Omega^* = 2A\sqrt{pg} = 0.8944$ the particle is oscillating in a sinusoidal manner between x_{min} and x_{max}, with $x_{max} = A$ and $x_{max} - x_{min}$ decreasing as Ω is approaching Ω^* (upper panel in Figure 6). At $\Omega = \Omega^*$, the particle is in relative equilibrium with respect to the tube (middle panel in Figure 6). Finally, for $\Omega > \Omega^*$ one observes an oscillation between $x_{min} = A$ and x_{max}, with $x_{max} - x_{min}$ increasing with Ω (lower panel in Figure 6).

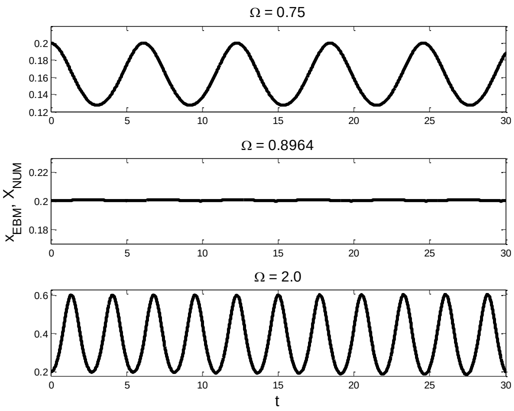

Figure 6. Dependence on time of the numerical solution for equation (9) if p = 0.5, A = 0.2 and $\Omega \in \{0.75, 0.8964, 2\}$.

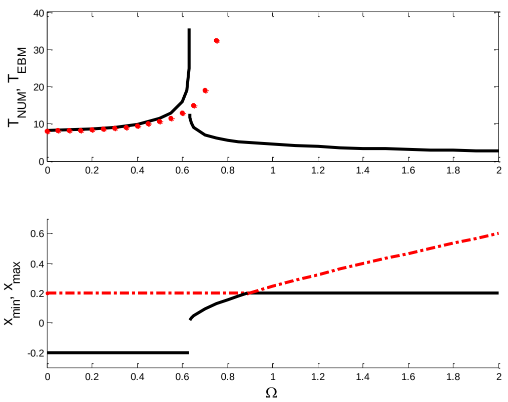

Figure 7. The dependencies $T_{NUM}(\Omega)$ and $T_{EBM}(\Omega)$ (upper panel) and the extreme positions of the particle during oscillation (lower panel) if p = 0.5 and A = 0.2.

An overview of the numerical solution's behaviour with increasing Ω is presented in Figure 7. In the upper panel are represented the dependencies $T_{NUM}(\Omega)$ and $T_{EBM}(\Omega)$. As stated before, the domain of definition for the second is reduced to $\Omega \le \sqrt{0.6}$ and the two periods are close enough just for $\Omega \le 0.5$. The faster the tube rotates, the slower the particle moves inside the tube, between $x_{min} = -A$ and $x_{max} = A$ (see the lower panel in Figure 7). T_{NUM} increases from 8.305 s for $\Omega = 0$ to 35.601 s at $\Omega = 0.632$. Between $\Omega = 0.632$ and $\Omega = 0.633$, there is a bifurcation in the motion's appearance. For few periods the particle changes the centre of oscillation but after some time it returns to $x_{REP} = 0$. After this transition region, it follows the three possibilities described in Figure 6, when the particle oscillates around a new centre with a period decreasing slower and slower with Ω. Excepting the small bifurcation area, the law

$$x(t) = a + b\cos\left(\frac{2\pi}{T_{NUM}}t\right), a = \frac{x_{max} + x_{min}}{2}, b = \frac{x_{max} - x_{min}}{2}$$

depicts fairly well the particle's motion inside the tube. Obviously, for small Ω one has a = 0 and b = A.

The same behaviour was noticed for other combinations (p, A, n) with the only difference that for larger p and A the sine wave was replaced by square wave.

4.3. $f(x) = p(1 - \cos x), x \in (-\pi, \pi), p > 0$

The energy balance method yields the following expression for the oscillation's frequency

$$\omega_{EBM} = \frac{2}{A}\sqrt{\frac{pg\left(\cos(A/\sqrt{2}) - \cos A\right) - \Omega^2 A^2/4}{1 + p^2\sin^2\left(A/\sqrt{2}\right)}} \quad (14)$$

Table 2. The periods T_{NUM}, T_{EBM} and relative error ε (%) between them for different p, A and Ω

	p = 0.5					
Ω	A = 0.2			A = 1.0		
	T_{NUM}	T_{EBM}	ε	T_{NUM}	T_{EBM}	ε
0.5	2.901	2.898	0.10	3.235	3.243	0.25
1.0	3.162	3.159	0.09	3.574	3.583	0.25
1.5	3.819	3.816	0.08	4.501	4.507	0.13
2.0	6.385	6.379	0.09	10.991	10.461	4.82
	p = 1.0					
Ω	A = 0.2			A = 1.0		
	T_{NUM}	T_{EBM}	ε	T_{NUM}	T_{EBM}	ε
0.5	2.037	2.038	0.05	2.523	2.563	1.59
1.0	2.120	2.121	0.05	2.643	2.683	1.51
2.0	2.600	2.601	0.04	3.372	3.421	1.45
2.9	6.507	6.510	0.05	16.805	12.033	28.4
	p = 2.0					
Ω	A = 0.2			A = 1.0		
	T_{NUM}	T_{EBM}	ε	T_{NUM}	T_{EBM}	ε
1.0	1.503	1.502	0.06	2.418	2.523	4.34
2.0	1.638	1.637	0.06	2.674	2.794	4.49
3.0	1.979	1.977	0.10	3.375	3.514	4.12
4.0	3.308	3.306	0.06	8.479	8.156	3.81

The numerical simulations reveal that the particle oscillates only around $x_{REP}=0$, no matter the triplet (p, A, Ω) is chosen for which the expression (14) makes sense. The relative error between the periods provided by numerical and energy balance method solutions behaves similarly to the quadratic parabola case. Some representative values are included in Table 2. The error ε grows toward unacceptable values only if Ω approaches the upper limit Ω^* and/ or for large amplitudes *A*.

5. CONCLUSIONS

In the paper, the highly nonlinear dynamics of a particle inside a rotating U-shaped frictionless thin tube is investigated both numerically and analitically via the energy balance method. First, the general equation of motion is established and then it is particularized for quadratic and quartic parabola as well as for the cosine curve. The main conclusions of the study are:

a) The results of numerical and analytical solutions for quadratic parabola and cosine curve are coincidence for small oscillation amplitudes but differ with one to ten percent for large amplitudes. The particle oscillates simetrically around the minimum point. The two solutions exist up to a critical alue of the tube's rotation frequency and then becomes unbounded or complex,

b) The explicit expression for oscillation frequency as a function of curve parameter, oscillation amplitude and rotation frequency is provided by a simple calculation and the role of each parameter can be easily investigated.

c) The motion inside a quartic parabola-like tube is characterized by three relative equilibrium points. For small rotation frequency the particle oscillates around the minimum point of the curve and enery balance method gives acceptable results. After a transition area, where the two solutions differ by tens of percent, the numerical solution continues to exist and shows that the particle oscillates in a sinusoidal manner around another relative equilibrium point. The oscillation amplitude increases with the rotation frequency while the oscillation period has a reverse behavior.

6. APPENDIX

Let us consider a thin frictionless tube with the shape of a curve $y=f(x)$ and a small ball M of mass *m* inside the tube in the location $(x, f(x))$. The tube is rotating with constant angular velocity Ω around *Oy* axis (see Figure A1). The ball's motion takes place according to the Newton's law for relative motion

$$m\vec{a}_r=\vec{R}+\vec{F}_t+\vec{F}_C \qquad \text{(A1)}$$

where $\vec{a}_r, \vec{R}, \vec{F}_t$ and $\vec{F}_C$ denote the vectors relative acceleration, resulting force of external and connection forces (here, the weight $m\vec{g}$ and normal reaction components $\vec{N}_1$ and $\vec{N}_2$), transport and Coriolis forces, respectively [11, 12]. The last two are given by $\vec{F}_t=-m\vec{a}_t$ and $\vec{F}_C=-m\vec{a}_C=-2m\left(\vec{\Omega}\times\vec{v}_r\right)$.

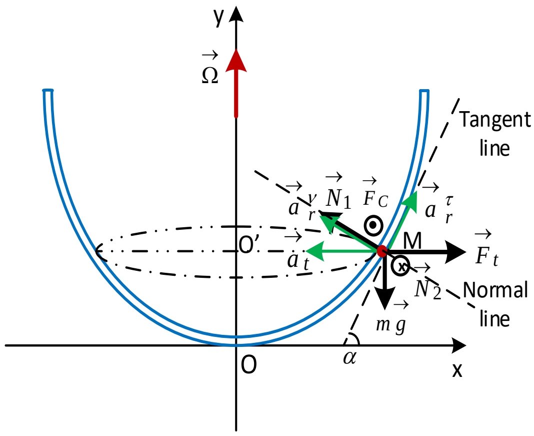

Figure A1. Distribution of forces and accelerations on the small ball M moving inside the rotating tube

Because Ω is constant, transport acceleration $\vec{a}_t$ is directed on the line MO', from M to O', and has the magnitude $a_t=MO'\Omega^2=x\Omega^2$.

Consequently, $F_t=ma_t=\ mx\Omega^2$. The vectors angular velocity $\vec{\Omega}$ and relative velocity $\vec{v}_r$ belong to the plane (OO'M), so the force $\vec{F}_C$ is perpendicular onto this plane (exit from it). Its magnitude is

$$F_C=2m\Omega v_r\sin\left(90^0-\alpha\right)$$

with v_r the relative velocity and α the angle between the tangent line and Ox. By projecting the equation (A1) on the tangent line, normal line and on the perpendicular to the curve plane, one has:

$$\begin{cases} ma_r^{\tau}=-mg\sin\alpha+F_t\cos\alpha \\ ma_r^{\nu}=N_1-mg\cos\alpha-F_t\sin\alpha \\ 0=N_2-F_C \end{cases} \qquad \text{(A2)}$$

Let $s(x)=\int_0^x\sqrt{1+(f'(\xi))^2}\,d\xi$ be the length of curve arc OM. Then

$$\dot{s}=\frac{ds}{dt}=\frac{ds}{dx}\cdot\frac{dx}{dt}=\sqrt{1+(f'(x))^2}\cdot\dot{x}$$

$$\ddot{s}=\frac{d}{dt}\left(\dot{s}\right)=\frac{f'(x)f''(x)}{\sqrt{1+(f'(x))^2}}\cdot\dot{x}^2+\sqrt{1+(f'(x))^2}\cdot\ddot{x}\,.$$

Additionally, $\tan\alpha=f'(x)$ so $\sin\alpha=\dfrac{f'(x)}{\sqrt{1+(f'(x))^2}}$ and $\cos\alpha=\dfrac{1}{\sqrt{1+(f'(x))^2}}$. Observing that $a_r^{\tau}=\ddot{s}$, the equation describing the ball's motion inside the tube (the first equation A2) is rewritten as

$$\left[1+\left(f'(x)\right)^2\right]\ddot{x}+f'(x)\,f''(x)\,\dot{x}^2=\Omega^2 x-g\,f'(x) \quad \text{(A3)}$$

Noting that $v_r=\dot{s}$ and $a_r^v=\frac{v_r^2}{\rho}$, with ρ being the radius of curvature at position M, the other equations (A2) allow us to determine the components N_1 and N_2 of normal reaction, $\vec{N}=\vec{N}_1+\vec{N}_2$.

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